The exact distribution of the Weighted Convolution of two Gamma distributions

La distribuzione esatta di una Convoluzione ponderata di due distribuzioni Gamma

Francesca Di Salvo

Dipartimento di Scienze Statistiche e Matematiche, Università degli studi di Palermo
e-mail: disalvo@dssm.unipa.it

Riassunto: Si considera una rappresentazione della funzione di densità di probabilità di una Convoluzione ponderata di distribuzioni Gamma, in cui una funzione ipergeometrica confluente descrive come le differenze tra i parametri di scala delle componenti determinino allontanamenti da una densità Gamma. Si considera il caso specifico di una convoluzione di due variabili Gamma per mostrare come al vantaggio interpretativo si aggiunga la possibilità di derivare in forma esplicita e computazionalmente semplice, espressioni della funzione di ripartizione e dei momenti. Alcune rappresentazioni grafiche mostrano gli effetti dei parametri sul comportamento della distribuzione. Si mostra inoltre la relazione tra tale distribuzione ed il sistema delle distribuzioni di Bessel.

Keywords: Gamma Convolution, hypergeometric functions, Bessel distributions.

1. Introduction

Let $\mathbf{X} = (X_1, \ldots, X_k)^T$ be a random vector of independent Gamma r.v.'s, $X_i \sim G(\alpha_i, \lambda)$. Given a vector $\mathbf{w} = (w_1, w_2, \ldots, w_k)^T$ of real and non random weights, we define the Weighted Gamma Convolution Y as the linear trasformation $Y = \sum_{i=1}^{k} w_i X_i$.
In Di Salvo (2005), the distribution of Y is characterized as the product between a Gamma density and a confluent hypergeometric function:

$$f_Y(y) = \frac{\left(\frac{\lambda}{w_k}\right)^{\alpha^*} y^{\alpha^*-1} e^{-y\frac{\lambda}{w_k}}}{\Gamma(\alpha^*)} \prod_{i=1}^{k-1} \left(\frac{w_k}{w_i}\right)^{\alpha_i} \Phi^{[k-1]}(\alpha^{[k-1]}; \alpha^*; y\mathbf{s}^{[k-1]}) \tag{1}$$

where

$$\alpha^* = \sum_{i=1}^{k} \alpha_i \qquad \mathbf{z}^{[n]} = (z_1, \ldots, z_n)^T \qquad w_k = \min_i \{w_i\} \qquad s_i = \lambda\left(\frac{1}{w_k} - \frac{1}{w_i}\right)$$

and $\Phi^{[k-1]}$is the confluent form of the fourth Lauricella function. The factor:

$$\prod_{i=1}^{k-1} \left(\frac{w_k}{w_i}\right)^{\alpha_i} \Phi^{[k-1]}(\alpha^{[k-1]}; \alpha^*; y\mathbf{s}^{[k-1]})$$

describes how differences in the weights, and hence in the scale parameters of the Gamma variables $w_i X_i$, produce a deviation from a Gamma density $G(\alpha^*, \lambda/w_k)$. When the weights are all equal, we have the particular case of a sum of k i.i.d Gamma variables, for

which the factor reduces to 1 and the density (1) reduces to a Gamma density.
Convolution of Gamma distributions are of interest in various field of applications: input-output or storage models (Mathai, 1982), in problems such as waiting times in queueing theory, stochastic processes (Sim, 1992), modelling distribution of composite sampling (Di Salvo, Lovison, 1999), in the evaluation of aggregate economic risk of portfolios (H$\ddot{u}$rlimann, 2001). Althought previous representation are given in literature for the distribution of Y, definition (1) offers an explicit and computable form for p.d.f., moments and c.d.f.. However efficient tools to evaluate exactly these functions require computational efforts and are subject of a forthcoming paper.
The object of this paper is to show some interesting results for the case of convolution of two gamma distributions: for this particular case, the density (1) allows expressions involving some special functions commonly implemented in standard statistical software; for this reason it may be considered a first good example of the general result.

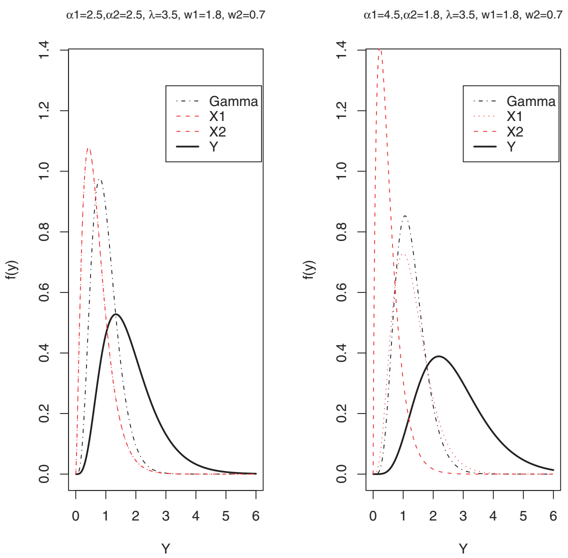
2. Representing the distribution of Y for $k = 2$

Setting $k = 2$, with substantial advantage from computational point of view, in the p.d.f. of $Y = w_1X_1 + w_2X_2$, the function $\Phi^{[k-1]}$ reduces to the Kummer's confluent hypergeometric function ${}_1F_1$ and one gets:

$$f_Y(y) = \left(\frac{w_2}{w_1}\right)^{\alpha_1} \frac{\left(\frac{\lambda}{w_2}\right)^{\alpha^*} y^{\alpha^*-1}\mathrm{e}^{-y\frac{\lambda}{w_2}}}{\Gamma(\alpha^*)} {}_1F_1\left(\alpha_1; \alpha^*; y\frac{\lambda}{w_2}\left(1 - \frac{w_2}{w_1}\right)\right) \tag{2}$$

Figure 1 shows the effects of the differences between weights, w_i, $i = 1, 2$, in the behaviour of the density:

Figure 1: *Examples of densities with equal (left) and different (right) shape parameters, compared with the gamma part (dashed-dots line) appearing in (2).*



In figure 1, the distance between $f_Y(y)$ and the Gamma density (dashed-dots line) is due to difference between weights. Simple expressions for the c.d.f. and moments of the distribution are then obtained as functions of the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ and of its incomplete version ${}_{[z]2}F_1(a, b; c; d)$ (cfr. Exton, 1976):

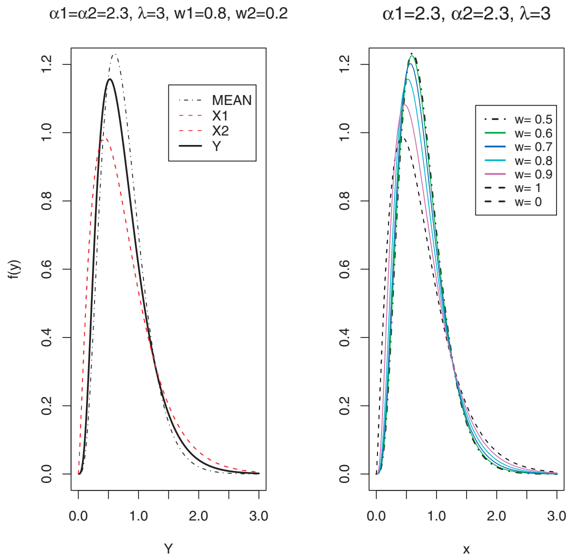
$$F_Y(y) = \left(\frac{w_2}{w_1}\right)^{\alpha_1} {}_{[y\frac{\lambda}{w_2}]2}F_1\left(\alpha^*, \alpha_1; \alpha^*; \left(1-\frac{w_2}{w_1}\right)\right) \tag{3}$$

$$E\left[Y^m\right] = \frac{(\alpha^*)_m}{(\lambda/w_2)^m}\left(\frac{w_2}{w_1}\right)^{\alpha_1} {}_2F_1\left(\alpha^*+m, \alpha_1; \alpha^*; \left(1-\frac{w_2}{w_1}\right)\right) \tag{4}$$

3. Relationships with the system of Bessel function distributions

Now we consider the weighted mean $Y = wX_1 + (1-w)X_2$, with $0 < w < 1$ and both X_i having the same shape parameter α. When w vary in the range $[0, 1]$ a family of distribution curves are described for Y, all of them lying between the density of the X_i, that's a $G(\alpha, \lambda)$, and a $G(2\alpha, 2\lambda)$, the density of the mean corresponding to $w_i = 0.5$; such behaviour is performed in the following graphics:

Figure 2: *Densities of Y for w=0.8 (left) and with w varying in [0,1](right).*



It is possible to demonstrate the equivalence of the density (2) with the first distribution of the system of Bessel function distributions (McKay, 1932):

$$f_Y(y) = \frac{|\,1-c^2\,|^{m+\frac{1}{2}}}{\sqrt{\pi}2^m b^{m+1}\Gamma(m+\frac{1}{2})} e^{-y\frac{c}{b}} \,|\, y \,|^m \,\pi I_m\left(\frac{y}{b}\right) \tag{5}$$

where $|\, c \,|> 1$, $b > 0$. To show the identity of (5) and (2), we substitute in (5):

$$b = 2\frac{(1-w)w}{\lambda\left[w-(1-w)\right]} \qquad c = \frac{1}{\left[w-(1-w)\right]} \qquad m = \alpha - \frac{1}{2} \tag{6}$$

From $(1-w) = min\{w, 1-w\}$ and $w < 1$, follows $\mid c \mid > 1$, $b > 0$. The (5) becomes:

$$\begin{aligned} f_Y(y) &= \frac{\sqrt{\pi}}{2^{\alpha}\Gamma(\alpha)}\left(1-\frac{1}{[w(1-w)]^2}\right)^{\alpha}\left(\frac{w-(1-w)}{w(1-w)}\right)^{\alpha+\frac{1}{2}}\lambda^{\alpha+\frac{1}{2}}\times \\ & \quad y^{\alpha-\frac{1}{2}}e^{\left[-\frac{y}{2}\left(\frac{\lambda}{1-w}+\frac{\lambda}{w}\right)\right]}\mathrm{I}_{(\alpha-\frac{1}{2})}\left(\frac{y}{2}\left(\frac{\lambda}{1-w}-\frac{\lambda}{w}\right)\right) \end{aligned} \tag{7}$$

where $I_{\nu}(z)$ is the modified Bessel function of first type; through the relation:

$$I_{\nu}(z) = \frac{\left(\frac{1}{2}z\right)^{\nu}\mathrm{e}^{-z}}{\Gamma(\nu+1)}{}_1F_1(\nu+\frac{1}{2}, 2\nu+1, 2z) \tag{8}$$

replacing $\mathrm{I}_{(\alpha-\frac{1}{2})}\left(\frac{y}{2}\left(\frac{\lambda}{1-w}-\frac{\lambda}{w}\right)\right)$ in (7), after semplifications we get the p.d.f. (2).

4. Discussion

The p.d.f. of a weighted convolution of two Gamma variables is a gamma-type function, expressed in explicit form using the Kummer's function; particular types of distributions are obtained and some other known distributions are recovered for particular values of the parameters; more important such representation makes this model feasible from standard statistical software. As further developments, the result (2) is useful to working out also the distribution of convolution of two matrix Gamma variables. Tackling with computational aspects of the distribution of weighted Gamma convolutions with $k > 2$ is beyond the scope of this paper, but it is worth of further investigations, as they may find interesting applications in statistics and applied probability.

References

Abramowitz M., Stegun I.A. (1965) *Handbook of Mathematical Functions*, Dover, NY

Di Salvo F. (2005) The Distribution of the Weighted Sum of Gamma Random Variables. *submitted paper*.

Di Salvo F., Lovison G. (1999) Parametric inference on composite samples with random weights. *Working Papers GRASPA, www.graspa.org*

Erdèlyi A. (editor), *Higher trascendental functions. vol.1*, McGraw-Hill Book Company.

Exton H., (1976). *Multiple Hypergeometric functions and applications*. Ellis Horwood.

Hurlimann W., (2001) Analytical evaluation of economic risk capital for portfolios of Gamma risks. *ASTIN Bulletin* , 31, n.1, 107–122

Johnson N.L., Kotz S., Balakhrishnan N. ,1995a. *Univariate continuous distributions*, Vol.1, 2nd Wiley New York.

Mathai A.M., 1982. Storage capacity of a dam with Gamma type inputs. *Annals of Institute of Statistics and Mathematics* , 34, 591–597.

Mathai A.M., 1993. *A Handbook of Generalized Special Function for Statistical and Physical Sciences*. Oxford Science Publications

McKay A.T., (1932). A Bessel function distribution. *Biometrika*, 24, 39–44

Sim C.H., 1992. Point processes with correlated Gamma interarrival times. *Statistics & Probability Letters*, 15, 135–141.

Slater L.J., 1960. *Confluent Hypergeometric functions*. Cambridge University Press.